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THE CONFORMLY INVARIANT THEORY OF ELASTICITY

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Abstract—Let us assume that two elastic solids M and N are topologically equivalent (homeomorphic) and S_M is a solution of some boundary problem for the solid M. Taking into account these facts we want to know when it is possible to build a solution S_N for N. There is no simple answer because it is necessary to take into account the metric properties of M and N. We want to modify (if it is required) the equations of the theory of elasticity and to assign the conformal weights to the field values so that the equations should be unchanged after the conformal change of metric. In other words, we work not with some appointed metric, but with the classes of metrics: any two metrics from the class are conformly equivalent. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Consider two elastic solids M and N. Let us assume that they are topologically equivalent (homeomorphic). Let S_M be a solution of some boundary problem for the solid M. Taking into account these facts we want to know when it is possible to build a solution S_N for N. We have not a simple answer because it is necessary to take into account the metric properties of M and N.

Formally, we consider the elastic solids M and N in the stressed state. Assume that they are Riemannian manifolds: g and h are the metrics on M and N, respectively. Suppose there is a diffeomorphism $\varphi : (N, \partial N) \to (M, \partial M)$; ∂—the designation of the border. M and N are not supposed to be isometric, therefore, in the general case $h \neq \varphi^* g$, where φ^* is the standard operation of the tensor's transfer under the diffeomorphism (the tensor $\varphi^* g$ is the so-called "pullback" tensor [see, e.g., Dubrovin *et al.* (1986)]). For example, consider the statically determined problem for M without body forces (we use the classical linear model of the elasticity):

$$D_g \sigma = 0, \quad \sigma|_{\partial M} = P \tag{1}$$

where P is the vector of external forces, σ is the vector-valued 2-form (in the language of exterior forms). The form σ can be written as:

$$\sigma = (\sigma^{ik} \varepsilon_{kmn} \, \mathrm{d}x^m \wedge \mathrm{d}x^n) \, \partial / \partial x^i \tag{2}$$

where (x^1, x^2, x^3) are some coordinates on M, ε_{kmn} is the Levi-Civita tensor: $\pm(\det(g))^{1/2}$ or 0; $\det(g)$ is the determinant of the metric g; σ^{ik} is the Cauchy tensor of stresses. D_g is the exterior differentiation corresponding to the metric g:

$$D_g \sigma = (\nabla_k \sigma^{ik} (\det(g))^{1/2} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3) \, \partial / \partial x^i.$$

If σ is the solution for eqn (1), then the 2-form $\varphi^* \sigma$ is consistently determined on N. But generally speaking, the relation

$$D_h \varphi^* \sigma = 0 \tag{3}$$

is not correct because $D_h \neq D_{\varphi^* g}$. The adequate situation also occurs in the other problems of the elasticity theory: we can not apply φ to transfer the solutions from M to N.

Suppose there is a diffeomorphism under which g and h are conformly equivalent, i.e., $h = \lambda^2 \varphi^* g$, λ—some function, $\lambda \neq 0$. For example, consider a sphere without the northern pole and a plane, then φ is the stereographic projection. Moreover, if M and N are two-dimensional then φ always exists locally (the theorem on the isothermic coordinates). In the two-dimensional case we have $\varphi^* g = \lambda^2 g$, where φ is any (bi)holomorphic function. In three dimensions, φ is motion, stretch, inversion and their composition. These transformations generate the group which is isomorphic to the group $O(4, 1)$, the global conformal group of sphere is $SL(2, C)$ [see Dubrovin *et al.* (1986)].

We want to know when $D_h \varphi^* \sigma = 0$ if $D_g \sigma = 0$. The answer appears to be complex: σ^{ik} should be the conformal density of weight $(-n-2)$ and the trace of tensor of stresses should be zero:

$$\nabla'_a \sigma'^{ca} = \lambda^N (\nabla_a \sigma^{ca} + \gamma_a \sigma^{ca}(N+n+2) - \gamma^c g_{ka} \sigma^{ka}) \tag{4}$$

where $\sigma'^{ca} = \lambda^N \sigma^{ca}$, $\gamma_a = \lambda^{-1} \nabla_a \lambda$; ∇'_a, ∇_a are the covariant derivatives corresponding to h and g, respectively; n is the dimension of M [see Penrose and Rindler (1984) or Birrell and Davies (1982) for the notation].

For example, it is well known (the isotropic case): if the plane strain occurs without change of volume then the tensor of stresses is traceless [see, e.g., Hahn (1985)]. Also it is known that in problems of torsion the trace of the tensor of stresses equals zero. Consequently, the equations of equilibrium are conformly invariant (c.i.): $\nabla'_a \sigma'^{ca} = \lambda^N \nabla_a \sigma^{ca}$. And $\varphi^* \sigma$ is the solution for (3).

Note that the classical Kilosov-Mushelishvilli's method (which uses the complex function of stresses and the conformal transformations) can not be applied to the two-dimensional case because the biharmonic operator is not c.i. (see below, Section 4).

Using the above arguments we wish to construct the conformly invariant theory of elasticity. We want to modify (if it is required) the equations of the theory of elasticity and to assign conformal weights to the field values so the equations should be unchanged after the conformal change of the metric. In other words, we work not with some appointed metric, but with the classes of metrics: any two metrics from the class are conformly equivalent. For the 2-surface these classes (the conformal structures) are numbered by means of a Teichmüller space and a moduli space. The moduli space is a factor-space of the Teichmüller space after action of the modular group $SL(2, Z)$ [see, e.g., Besse (1987)]. In particular, if the surface is homeomorphic to the standard sphere S^2, then all metric structures are conformly equivalent and the moduli space is trivial (the so-called "Riemann theorem": if N is diffeomorphic to S^2 $(=M)$, then h and g are conformly equivalent). So, let us suppose that the surface M is an elastic solid, M is homeomorphic to the sphere and the tensor of stresses has the trace which is equal to zero. Then the elastic effects will be described identically (within conformal corrections) for any homeomorphic surface N.

In the general case, for a Riemannian surface of genus $k = 0$ (a sphere), we have not an obvious formula for the conformal diffeomorphism between M and N. Moreover, for a surface of genus $k > 0$ (a sphere with k handles), the moduli space is non-trivial (there are "many" conformly non-equivalent metric structures). Generally speaking, the conformal structures of Riemannian surface of genus $k > 2$ are described by moduli. The quantity of moduli equals $3k-3$. For example, if the surface M is a torus, then the Teichmüller space is the upper (open) half-plane (on the complex plane) and the quantity of moduli (the quantity of conformly invariant complex parameters) equals unity. However there is an extensive class of two-surfaces, so-called minimal surfaces. For these surfaces all calculations can be done.

The traceless tensor of stresses is the limitation for the applications of the conformly invariant theory of elasticity. However if the tensor of stresses is a function of some parameter t: $\sigma(t)$, then there always exists the value $t = t_0$ under which trace $(\sigma(t_0)) = 0$. Indeed (Poincaré idea), consider the infinite dimensional space of the tensors of stresses Σ in "general position" (see Arnold (1978) for discussion). The set $\Pi = \{\sigma^{ik}|\text{trace}(\sigma^{ik}) = 0\}$ is "the surface" in Σ, i.e., Π has a codimensional unity. Since $\sigma(t)$ is "the curve" in Σ and $\sigma(t)$ is transversal to Π, it follows that $\sigma(t)$ crosses the surface Π.

2. BASIC EQUATIONS OF THE LINEAR THEORY OF ELASTICITY

Consider the basic relationships for M and N (without body forces and thermodynamic effects):

$$D_g\sigma = 0, \quad D_h\sigma' = 0 \tag{5}$$

$$2e_{ik} = g_{ik} - g^{\circ}_{ik}, \quad 2e'_{ik} = h_{ik} - h^{\circ}_{ik} \tag{6}$$

$$\sigma^{ik} = A^{ikmn}e_{mn}, \quad \sigma'^{ik} = A'^{ikmn}e'_{mn} \tag{7}$$

$$R^{\circ}_{ik} = 0, \quad R^{\circ\prime}_{ik} = 0 \tag{8}$$

$$R_{ik} = 0, \quad R'_{ik} = 0 \tag{9}$$

where g°_{ik}, h°_{ik} are the metrics on M and N before deformation; g_{ik}, h_{ik} are the metrics after deformation; R°_{ik}, $R^{\circ\prime}_{ik}$, R_{ik}, R'_{ik} are the Ricci tensors for these metrics, respectively. The metrics g and h are conformly equivalent: $h_{ik} = \lambda^2(\varphi^*g)_{ik}$ for some diffeomorphism φ. We want to know when eqns (5)–(9) are c.i.

2.1. *The equations of equilibrium*

The equations of equilibrium (5) are conformly invariant if the trace of the tensor of stresses equals zero and σ^{ik} is the conformal density of weight $(-n-2)$ (see eqn (4)):

$$\nabla'_a\sigma'^{ab} = \nabla_a\sigma'^{ab} + \Lambda^a_{ah}\sigma'^{hb} + \Lambda^b_{ah}\sigma'^{ah} = \lambda^Q(\nabla_a\sigma^{ab} + Q\gamma_a\sigma^{ab} + \gamma_a\sigma^{ab} + n\gamma_h\sigma^{hb} +$$
$$\gamma_a\sigma^{ab} + \gamma_h\sigma^{hb} - \gamma^a\sigma^b_a - \gamma^b g_{kn}\sigma^{kn}) = \lambda^Q(\nabla_a\sigma^{ab} + \gamma_a\sigma^{ab}(Q+n+2) - \gamma^b g_{kn}\sigma^{kn}) \tag{10}$$

where

$$\Lambda^i_{jk} = \Lambda^i_{kj} = \gamma_j\delta^i_k + \gamma_k\delta^i_j - \gamma^i g_{jk}.$$

2.2. *Hooke law*

If the equations of equilibrium are conformly invariant, then the sum of weights A'^{ikmn} and e'_{mn} equals $(-n-2)$. Let A'^{ikmn} be the conformal density of the weight Q: $A'^{ikmn} = \lambda^Q A^{ikmn}$ then, from eqns (5)–(7):

$$A^{ikmn}\lambda^Q(h_{mn} - h^{\circ}_{mn}) = A^{ikmn}\lambda^{-(n+2)}(g_{mn} - g^{\circ}_{mn}). \tag{11}$$

For the ease of notation it is admitted: $T = \varphi^*T$, where T is a tensor. From eqns (6) and (11) we obtain under $Q = -(n+4)$:

$$e'_{ik} = \lambda^2 e_{ik}. \tag{12}$$

Under $Q = -(n+2)$:

$$e'_{ik} = e_{ik}. \tag{13}$$

In particular, if M is isotropic then in coordinates on N:

$$A^{ijkn} = \mu g^{ij} g^{kn} + \nu(g^{ik} g^{jn} + g^{in} g^{jk}) \tag{14}$$

$$A'^{ijkm} = \lambda^{-n}(\mu h^{ij} h^{km} + \nu(h^{ik} h^{jm} + h^{im} h^{jk})), \quad \text{under } Q = -(n+4) \tag{15}$$

$$A'^{ijkm} = \lambda^{-(n-2)}(\mu h^{ij} h^{km} + \nu(h^{ik} h^{jm} + h^{im} h^{jk})), \quad \text{under } Q = -(n+2) \tag{16}$$

where μ, ν are the Lame's coefficients.

2.3. *Ricci tensor*

If the metric h is not flat then the equations of compatibility of strains (8)–(9) are not c.i. It follows from the law of transformation of Ricci curvature under the conformal change of the metric [see Penrose and Rindler (1984); Birrell and Davies (1982); Besse (1987)]:

$$h^{ik} R'_{kj} = R'^{i}_{j} = \lambda^{-2}[R^i_j - (n-2)(\gamma^i \gamma_j - \nabla^i \gamma_j) + \delta^i_j((n-2)\gamma^k \gamma_k + \nabla^k \gamma_k)]. \tag{17}$$

The scalar curvature transforms as:

$$R' = \lambda^{-2}[R + \gamma^k \gamma_k(n(n-2) - (n-2)) + \nabla^k \gamma_k(2n-2)] \tag{18}$$

i.e., $R'_{ik} \neq 0$, $R' \neq 0$ (in the general case). We interpret "the superfluous terms" into (17) as the tensor of incompatibility of strains B, which describes the distribution of discontinuities in a continuous media [see Kröner (1958), Kröner (1961)]. Under $n = 3$ we have: $B^i_j = h^{ik} B_{kj} = \lambda^{-2}[\gamma^i \gamma_j - \nabla^i \gamma_j + \delta^i_j(\gamma^k \gamma_k + \nabla^k \gamma_k)]$, under $n = 2$ the Ricci tensor is defined by its trace, therefore we may use the function: $B = \lambda^{-2} \nabla^k \nabla_k \ln \lambda$. So, if we have a solution of the problem for M: $D_g \sigma = 0$, $\sigma^{ik} = A^{ikmn} e_{mn}$, $2e_{mn} = (g_{mn} - g^0_{mn})$, $R_{ij} = 0$; $g_{ik} \sigma^{ik} = 0$ then on N we obtain automatically: $D_h \sigma' = 0$, $\sigma'^{ik} = A'^{ikmn} e'_{mn}$, $2e'_{mn} = (h_{mn} - h^0_{mn})$, $R'_{ij} = B_{ij}$. Where $h_{mn} = \lambda^2 g_{mn}$, $A'^{ikmn} = \lambda^Q A^{ikmn}$, $e'_{mn} = \lambda^P e_{mn}$, $\sigma'^{ik} = \lambda^{-(n+2)} \sigma^{ik}$; $Q = -(n+4)$ or $Q = -(n+2)$, $P = 2$ or $P = 0$.

2.3.1. *The continuum theory of defects.* It is well known that in the Kröner theory:

$$\nabla \times e \times \nabla = \eta \tag{19}$$

where e is the tensor of strains, $\eta = (\alpha \times \nabla)^s$ is the tensor of incompatibility, $(\ldots)^s$ is the symmetrization, α is the density of dislocation. The formula (19) follows from the equation $R_{(ik)}(\Gamma) = 0$, where $R_{(ik)}(\Gamma)$ is the symmetrical part of the Ricci curvature of the non-symmetrical connection Γ (the connection with a torsion), i.e., in the common case $R_{ik}(\Gamma) \neq R_{ki}(\Gamma)$. We have the common formula for the connection Γ (g_{ik} is some Riemannian metric) [see Schrödinger (1950)]:

$$\Gamma^s_{ik} = \begin{Bmatrix} s \\ ik \end{Bmatrix} - \frac{1}{2} g^{sl} g_{im} \Gamma^m_{[kl]} - \frac{1}{2} g^{sl} g_{km} \Gamma^m_{[il]} + \Gamma^s_{[ik]} = \begin{Bmatrix} s \\ ik \end{Bmatrix} + g^{sl} T_{lik} + \Gamma^s_{[ik]} = \begin{Bmatrix} s \\ ik \end{Bmatrix} + K^s_{ik},$$

where

$$g^{sl} T_{lik} = -\frac{1}{2} g^{sl} g_{im} \Gamma^m_{[kl]} - \frac{1}{2} g^{sl} g_{km} \Gamma^m_{[il]} = g^{sl} T_{lki}, \quad K^s_{ik} = g^{sl} T_{lik} + \Gamma^s_{[ik]}, \quad \text{where } \begin{Bmatrix} s \\ ik \end{Bmatrix}$$

is the Christoffel connection,

$$\Gamma^s_{[ik]} = \frac{1}{2}(\Gamma^s_{ik} - \Gamma^s_{ki})$$

is the skewsymmetric tensor. Note that the term $g^{sl}T_{lik}$ is c.i. The Ricci curvature of the connection Γ^s_{ik} can be written as:

$$R_{ik}(\Gamma) = R_{ik}(\{\ \}) + \nabla_s K^s_{ik} - \nabla_k K^s_{is} + K^s_{sp}K^p_{ik} - K^s_{kp}K^p_{is}, \tag{20}$$

$R_{ik}(\{\ \})$ is the (symmetrical) Ricci tensor of the Christoffel connection, ∇_k is the "ordinary" covariant derivative corresponding to g_{ik}. If $h_{ik} = \lambda^2 g_{ik}$ and g_{ik} are flat metrics then:

$$\begin{aligned}
&(n-2)(\gamma_i\gamma_k - \nabla_i\gamma_k) + g_{ik}((n-2)(\gamma^s\gamma_s + \nabla^s\gamma_s)) + \\
&+ (\nabla'_s K^s_{ik} - \nabla'_k K^s_{is} + K^s_{sp}K^p_{ik} - K^s_{kp}K^p_{is})_{(ik)} \\
&= (n-2)(\gamma_i\gamma_k - \nabla_i\gamma_k) + g_{ik}((n-2)(\gamma^s\gamma_s + \nabla^s\gamma_s)) + \\
&+ (\nabla_s K^s_{ik} - \nabla_k K^s_{is} + K^s_{sp}K^p_{ik} - K^s_{kp}K^p_{is} + \Lambda^s_{hs}K^h_{ik} - \Lambda^s_{hi}K^h_{sk} - \Lambda^s_{kh}K^h_{is} + \Lambda^s_{ki}K^h_{sh})_{(ik)} = 0
\end{aligned} \tag{21}$$

where ∇'_k is the covariant derivative corresponding to h_{ik}. The eqn (21) connects the conformal anomalies and the torsion. The eqn (21) follows from (17) and (20) (under $n = 2$ we use the relation $R_{ac} = R/2g_{ac}$).

2.4. *The displacement vector*

It is known that in the classical linear theory $R^\circ_{ik} = 0$ and $R_{ik} = 0$, so

$$2e_{ik} = (\nabla_i u_k + \nabla_k u_i + \nabla_i u_h \nabla_k u^h) \quad \text{(in Lagrange coordinates)} \tag{22}$$

where u_k is the displacement vector. In the case of small strains

$$2e_{ik} = (\nabla_i u_k + \nabla_k u_i). \tag{23}$$

The eqns (22) and (23) are not c.i. In particular, for the case (23) we have:

$$e'_{ik} = \lambda^Q(e_{ik} + (Q-2)\gamma_i u_k + (Q-2)\gamma_k u_i + \gamma^h g_{ik} u_h),$$

where it is assumed that u_k is the conformal density of weight Q.

Therefore the displacement vector (in contrast to the tensor of strains) is not conformal density. Then, after the replacement $g \to h = \lambda^2 g$, we calculate by means of the Cesaro formula:

$$\begin{aligned}
u'_i &= u^{0\prime}_i + \omega^{0\prime}_{ik}x^k + \int_\Gamma (e'_{ik} - (\nabla'_j e'_{ik} - \nabla'_i e'_{jk})x^j)\,dx^k \\
&= u^{0\prime}_i + \omega^{0\prime}_{ik}x^k + \int_\Gamma (e'_{ik} - (\nabla_j e'_{ik} - \nabla_i e'_{jk})x^j)\,dx^k - \int_\Gamma (\gamma_i e'_{jk} - \gamma_j e'_{ik} + \gamma^h g_{ik} e'_{jh} - \gamma^h g_{jk} e'_{ih})x^j\,dx^k
\end{aligned} \tag{24}$$

where $e'_{ik} = \lambda^Q e_{ik}$, Γ is a path, u^0_i, ω^0_{ik}, $u^{0\prime}_i$, $\omega^{0\prime}_{ik}$ are the primary displacements. It is important to understand that in eqn (24) h and h° are flat metrics. If $Q = 0$ (see eqn (13)) then we have $u'_k = u_k + \Delta_k$ from eqn (24), where Δ_k is "the anomalous displacements":

$$u'_i = u_i + (u^{0\prime}_i - u^0_i) + (\omega^{0\prime}_{ik} - \omega^0_{ik})x^k - \int_\Gamma (\gamma_i e_{jk} - \gamma_j e_{ik} + \gamma^h g_{ik} e_{jh} - \gamma^h g_{jk} e_{ih}) x^j \, dx^k. \quad (25)$$

2.4.1. *The important note.* Consider an alternative theory of strains. We define the displacement covector as the conformal density of weight 0 ($g_{ik} = \delta_{ik}$):

$$B_{ak} \, dx^a \equiv D(u_k) = (\partial_a u_k - \Lambda^h_{ak} u_h) \, dx^a.$$

Then B_{ki} is a non-integrable distortion, i.e., $B_{ak} \, dx^a \neq dH_k$, where H_k is some (covector-valued) 0-form. By analogy with the gauge theory of defects [see Cadic and Edelen (1983)] we consider the tensor Λ^h_{ki} as "the gauge field". After the replacement $g \to h = \lambda^2 g$ we have:

$$e'_{ik} = e_{ik} - 2(\gamma_i u_k + \gamma_k u_i) + \gamma^h g_{ik} u_h.$$

We define the tensor of the plastic strains as

$$e^P_{ik} = -2(\gamma_i u_k + \gamma_k u_i) + \gamma^h g_{ik} u_h.$$

Then, e'_{ik} is the tensor of the general strains. We have:

$$\eta^{pq} = \varepsilon^{pmk} \varepsilon^{qni} \nabla_m \nabla_n (-2(\gamma_i u_k + \gamma_k u_i) + \gamma^h g_{ik} u_h)$$

—the tensor of incompatibility;

$$\alpha^{pi} = \varepsilon^{pmk} \nabla_m (-2(\gamma^i u_k + \gamma_k u^i) + \gamma^h \delta^i_k u_h)$$

—the density of dislocation.

2.5. *The boundary conditions*

The diffeomorphism transfers points, tangent vectors and covectors. Consequently we may transfer the boundary conditions from M to N. After the conformal change of a metric normal vectors are transformed to collinear vectors. Therefore, if the field values are the conformal densities, then it's simple to transfer the boundary conditions from M to N.

2.5.1. *The case of external forces*: $\sigma|_{\partial M} = P$ (*i.e.*, $\sigma^{ik} n_k = P^i$). It is necessary to assign the conformal weights to the normal covector n and the vector of external forces so that: weight(σ) + weight(n) = weight(P).

For example, in the important two-dimensional case, where the boundary of M is the plane curve, we have

$$n = \left(-\frac{\dot{x}^2}{|\dot{x}|}, \frac{\dot{x}^1}{|\dot{x}|} \right) \to n' = \lambda^{-1} n$$

(n is the vector), where (x^1, x^2) are arbitrary conditions of M, $\dot{x} = dx/dt$, $|\dot{x}| = \sqrt{g_{ik} \dot{x}^i \dot{x}^k}$, t is a parameter on the curve; consequently, weight (P) = $(-4) - (-1) = (-3)$.

2.5.2. *The case of the displacement (co)vector* u_i. We know that the displacement vector is not the conformal density (see Section 2.4), therefore, in the common case, on N:

$u'_i|_{\partial N} \neq \lambda^Q u_i|_{\partial N}$; however under $Q = 0$ (see eqn (25))

$$u'_i|_{\partial N} = u_i|_{\partial N} + \Delta_i|_{\partial N}.$$

2.6. *The notes*

a) If $\sigma^{ik} = \sigma^{ik}(e_{mn}) = A^{ikmn}e_{mn} + B^{ikmnpq}e_{mn}e_{pq} + \cdots$ (the non-linear theory), then we have: weight $(B^{ikmnpq}) = \text{weight}(\sigma^{ik}) - 2(\text{weight}(e_{mn}))$.

b) Let h and g be the flat metrics, then the equation for the Lame's potential Φ: $\nabla^k\nabla_k\Phi = 0$ is c.i. This follows from the law of transformation of the harmonic operator. The function Φ has the conformal weight $(2-n)/2$:

$$\left(\nabla'^a\nabla'_a + \frac{1}{4}\frac{(n-2)}{(n-1)}R'\right)\Phi' = \lambda^{-(n+2)/2}\left(\nabla^a\nabla_a + \frac{1}{4}\frac{(n-2)}{(n-1)}R\right)\Phi.$$

c) If we consider: $\sigma^{ik} = R^{ik} - \frac{1}{2}Rg^{ik}$ [see Pauli (1958)], $n = 2$, then the equations of equilibrium (without body forces) are satisfied and they are c.i. because $g_{ik}\sigma^{ik} = 0$. But in this case the conformal weight of the tensor of stresses equals zero (see the formula (4), where the conformal weight equals (-4)).

d) Consider the case $Q = -(n+4)$ and $n = 2$: $\overset{\circ}{R}_{ik} = 0$, $\overset{\circ}{R}_{ik}{}' = 0$, $R_{ik} = 0$, $R'_{ik} = 0$. Then we obtain from $\overset{\circ}{h}_{ik} = \lambda^2\overset{\circ}{g}_{ik}$: the equations $(R = 0, R' = 0)$ and $(\Delta_{g^0}\ln\lambda = 0, \Delta_g\ln\lambda = 0)$ are equivalent; where Δ_{g^0}, Δ_g are the harmonic operators corresponding to $\overset{\circ}{g}_{ik}$ and g_{ik}, respectively. If $\overset{\circ}{g}_{ik} = g_{ik} + \varepsilon f_{ik} = \delta_{ik} + \varepsilon f_{ik}$ (the classical linear theory) then we have:

$$\Delta_{g^0}\ln\lambda = \Delta_g\ln\lambda + \varepsilon\left(\partial_i\left(\frac{\alpha}{2}\delta^{ij}\right)\partial_j\ln\lambda - \partial_i f^{ij}\partial_j\ln\lambda\right) + o(\varepsilon),$$

where $\varepsilon \to 0$, f_{ik} is some metric, $\alpha = f_{11} + f_{22}$.

e) If $Q = -(n+2)$ and $n = 2$ ($\overset{\circ}{R}_{ik} = 0$, $\overset{\circ}{R}_{ik}{}' = 0$, $R_{ik} = 0$, $R'_{ik} = 0$) then we have

$$\overset{\circ}{h}_{mn} = \lambda^2\overset{\circ}{g}_{mn} + \varepsilon(\lambda^2 - 1)f_{mn}, \quad \text{where } g_{ik} = \overset{\circ}{g}_{ik} + \varepsilon f_{ik}.$$

f) We consider only two alternatives under a choice of weight Q (formulae (12), (13)).

3. TWO-DIMENSIONAL PROBLEM (THE ISOTROPIC AND ANISOTROPIC CASES)

3.1. *The biharmonic operator*

Consider how the biharmonic equation $\Delta^2\psi = \nabla^b\nabla_b\nabla^a\nabla_a\psi = 0$ changes under the conformal change of the scale. Let ψ have the conformal weight P:

$$\begin{aligned}\Delta'^2\psi' = \nabla'^b\nabla'_b\nabla'^a\nabla'_a\psi' = \lambda^{P-4}[&P^2\{Q^2\gamma^b\gamma_b + 2Q\gamma^b\nabla_b + \nabla^b\nabla_b + Q\nabla^b\gamma_b\}\gamma^a\gamma_a\psi + \\ &+ 2P\{Q^2\gamma^b\gamma_b + 2Q\gamma^b\nabla_b + \nabla^b\nabla_b + Q\nabla^b\gamma_b\}\gamma^a\nabla_a\psi + \\ &+ P\{Q^2\gamma^b\gamma_b + 2Q\gamma^b\nabla_b + \nabla^b\nabla_b + Q\nabla^b\gamma_b\}\nabla^a\gamma_a\psi + \\ &+ \{Q^2\gamma^b\gamma_b + 2Q\gamma^b\nabla_b + Q\nabla^b\gamma_b\}\nabla^a\nabla_a\psi + \Delta^2\psi],\end{aligned}$$

where $Q = P - 2$. Under $P = 2$ we obtain:

$$\Delta'^2\psi' = \lambda^{-2}[\nabla^b\nabla_b\{4\gamma^a\gamma_a\psi + 4\gamma^a\nabla_a\psi + 2\nabla_a\gamma^a\psi\} + \Delta^2\psi] \tag{26}$$

under $P = 0$:

$$\Delta'^2\psi' = \lambda^{-4}[\{4\gamma^b\gamma_b - 4\gamma^b\nabla_b - 2\nabla_b\gamma^b\}\nabla^a\nabla_a\psi + \Delta^2\psi]. \tag{27}$$

Note that if g and h are flat metrics, then $\nabla_b\gamma^b = 0$ (see eqn (18)).

The conclusion: in the general case, the biharmonic operator is not c.i. under any choice of weight.

Nevertheless, the Erie's function (the function of stresses) F is harmonic. It follows from the two-dimensional equations of equilibrium, from the equations of compatibility and the formulas for the components of the stress tensor. We have ($g = \delta_{ik}\,dx^i\,dx^k = dx^2 + dy^2$):

$$\sigma^{11} = \frac{\partial^2 F}{\partial y^2}, \quad \sigma^{22} = \frac{\partial^2 F}{\partial x^2},$$

on the other hand (see eqn (10)) the trace of tensor of stresses is

$$\sigma = \sigma^{11} + \sigma^{22} = \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial x^2} = 0.$$

Consequently, the equation $\Delta(\sigma^{11} + \sigma^{22}) = 0$ is trivial and F is harmonic (in the isotropic case). Therefore the basic equation of the conformly invariant theory of elasticity (in the plane problem) is the harmonic equation. The Erie's function is the conformal density of weight zero (see Section 2.6 (b)).

In the orthotropic case we have:

$$\left(\frac{1}{E_1} - \frac{1}{E_2} - \frac{1}{\mu_{12}}\right)\frac{\partial^4 F}{\partial y^4} = 0$$

where E_i, μ_{12} are the technical constants. Taking into account that

$$\frac{\partial^2 F}{\partial y^2} = -\frac{\partial^2 F}{\partial x^2} \quad \text{and} \quad \left(\frac{1}{E_1} - \frac{1}{E_2} - \frac{1}{\mu_{12}}\right) \neq 0$$

(in the general case) we obtain:

$$\frac{\partial^4 F}{\partial y^4} = \frac{\partial^4 F}{\partial x^4} = \frac{\partial^4 F}{\partial y^2 \partial x^2} = 0.$$

It is easy to understand that F is also harmonic. If we consider the general anisotropy, then the function of stresses is also harmonic.

We introduce the function

$$\Phi : \Phi = F + i\Psi, \quad i = \sqrt{-1}, \quad \frac{\partial F}{\partial x} = \frac{\partial \Psi}{\partial y}, \quad \frac{\partial F}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

(the Cauchy-Riemann conditions), i.e., F and Ψ are conjugate. It is easy to show that (refer to [Hahn (1985)] for the notions):

$$\sigma^{11} = \frac{\partial^2 F}{\partial y^2} = -\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)^2 F(z,\bar{z}) = -\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)^2 \operatorname{Re}\Phi = -\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)^2 \frac{1}{2}(\Phi + \bar{\Phi})$$

$$= -\frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \bar{\Phi}}{\partial \bar{z}^2}\right) = -\frac{1}{2}(\Phi''(z) + \overline{\Phi''}(\bar{z}))$$

$$\sigma^{22} = \frac{\partial^2 F}{\partial x^2} = \frac{1}{2}(\Phi''(z) + \overline{\Phi''}(\bar{z}))$$

$$\sigma^{12} = \sigma^{21} = -\frac{\partial^2 F}{\partial x \partial y} = -i\frac{1}{2}(\Phi''(z) - \overline{\Phi''}(\bar{z})).$$

By analogy with the first and the second Kilosov's formulas we get:

$$\sigma_{xx} + \sigma_{yy} = 0$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2\overline{\Phi''}(\bar{z}).$$

In the isotropic case we have:

$$e_{xx} + e_{yy} = 0$$

$$e_{xx} - e_{yy} + 2ie_{xy} = -\frac{2}{E}(1+\nu)\overline{\Phi''}(\bar{z})$$

where E is the Young's modulus, ν is the Poisson's coefficient. The third Kolosov's formula is:

$$\frac{E}{(1+\nu)}D = -\overline{\Phi'}(\bar{z})$$

where $D = u_x + iu_y$ is the complete displacement vector. If the vector of stresses $p_x + ip_y$ is given as the function on the outline s, then we may write the boundary conditions as:

$$\overline{\Phi'}(\bar{z}) = -\int (p_y - ip_x)\,\mathrm{d}s + \text{const}$$

or: $\overline{\Phi'}(\bar{z}) = g_1 + ig_2$, where g_1 and g_2 are some functions. The stream of forces between points A and B can be written as:

$$(X + iY)_{AB} = \int_A^B (p_x + ip_y)\,\mathrm{d}s = -i[\overline{\Phi'}(\bar{z})]_A^B.$$

The moment of stream of forces with respect to the beginning of coordinates:

$$M_{AB} = [\operatorname{Re}(z2F'(z,\bar{z}))]_A^B = [\operatorname{Re}(z\Phi'(z))]_A^B.$$

If the displacement vector is given as $D^0 = u_x^0 + iu_y^0$, then:

$$\frac{E}{(1+\nu)} D^0 = -\overline{\Phi'}(\bar{z}).$$

3.2. *The theory of elasticity in a conformal gauge*

Let (x, y) denote isotermic coordinates on the surface (N, h): $h = ds^2 = e^{\varphi} g = e^{\varphi}(dx^2 + dy^2) = e^{\varphi} dz\, d\bar{z}$—"the conformal gauge" [see, e.g., Green *et al.* (1987)], where $z = x + iy$. We define the standard basis (x, y) and $(z, \bar{z})$:

$$\partial_+ = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \partial_- = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$dz = dx + i\,dy, \quad d\bar{z} = dx - i\,dy$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)$$

$$dx = \frac{1}{2}(dz + d\bar{z}), \quad dy = \frac{1}{2i}(dz - d\bar{z}).$$

For example, if in the coordinates $(x^1 = x, x^2 = y)$ we have some symmetrical traceless tensor $T_{ik}\, dx^i\, dx^j$, then in $(z, \bar{z})$ coordinates, only $T_{++} \neq 0$ and $T_{--} \neq 0$:

$$T = T_{ik}\, dx^i\, dx^k = T_{++}\, dz\, dz + T_{--}\, d\bar{z}\, d\bar{z} + 2T_{+-}\, dz\, d\bar{z} = (2T_{+-} + (T_{++} + T_{--}))\, dx\, dx$$
$$+ (2T_{+-} - (T_{++} + T_{--}))\, dy\, dy + 2i(T_{++} - T_{--})\, dx\, dy,$$

but:

$$(4T_{+-} + (T_{++} + T_{--}) - (T_{++} + T_{--})) = 4T_{+-} = 4T_{-+} = 0.$$

The metric $h_{ik} = e^{\varphi}\delta_{ik}$ has components:

$$h_{++} = h_{--} = 0, \quad h_{+-} = h_{-+} = \tfrac{1}{2}e^{\varphi}.$$

Moreover, h is the covariant constant tensor:

$$\nabla_+ h_{+-} = \nabla_+ h_{-+} = \nabla_- h_{+-} = \nabla_- h_{-+} = 0.$$

The equations of equilibrium are:

$$\nabla_+ \sigma'^{++} + \nabla_- \sigma'^{-+} = 0$$
$$\nabla_+ \sigma'^{+-} + \nabla_- \sigma'^{--} = 0.$$

Since only $\sigma^{++} \neq 0$ and $\sigma^{--} \neq 0$, it follows that

$$\nabla_+ \sigma'^{++} = (\partial_+ + 2\partial_+ \varphi)\sigma'^{++} = 0$$
$$\nabla_- \sigma'^{--} = (\partial_- + 2\partial_- \varphi)\sigma'^{--} = 0.$$

If $\sigma' = e^{-2\varphi}\sigma$, i.e., σ is the conformal density of weight (-4) (formula (10)), then:

$$\nabla_+\sigma'^{++} = e^{-2\varphi}\partial_+\sigma^{++} = 0$$

$$\nabla_-\sigma'^{--} = e^{-2\varphi}\partial_-\sigma^{--} = 0. \tag{28}$$

Now we can obtain the relations between real and complex coordinates:

$$\sigma_{11} = \sigma_{xx} = -\sigma_{yy} = (\sigma_{++} + \sigma_{--}),$$
$$\sigma_{12} = \sigma_{xy} = \sigma_{yx} = i(\sigma_{++} - \sigma_{--}).$$

From (28) we have that the component σ^{--} is a holomorphic function:

$$\partial_-\sigma^{--} = 0. \tag{29}$$

From eqn (29) it follows that $\sigma^{--} \to 0$ under $z \to 0$, but it is impossible if N is homeomorphic to a sphere. If N is a Riemannian surface of the genus one then the component σ^{--} can be only constant. In the general case, for a Riemannian surface of the genus $k > 2$ the space of solutions for eqn (29) has the dimension $(3k-3)$ (the number of zero modes). The number $(3k-3)$ is proportional to the Euler characteristic of Riemannian surface (the surface of genus k has the Euler characteristic $(2-2k)$). The relation between the topology of surface and the zero modes is the consequence the index theorem [see, e.g., Green *et al.* (1987)]. In our case the elliptical operator is square of operator ∇ (see a discussion in [Green *et al.* (1987)]).

The equations of compatibility of strains in the conformal gauge we may write as:

$$\partial_+\partial_-\varphi = 0.$$

3.3. *The shell theory*

We consider a non-trivial application of the conformly invariant theory of elasticity to the shell theory. If the surface of a shell has global isotermic coordinates (u, v), then we have a map from the domain $U \subseteq R^2$ to the surface, where a point $(u, v) \in U$. Consequently, the metric of the plane $g = du^2 + dv^2$ and the metric of the surface $h = e^{\psi}g = e^{\psi}(du^2 + dv^2)$ are conformly equivalent. If the shell is a minimal surface then we can write complete formulae. We use a Weierstrass representation [see, e.g., Tuzilin and Fomenko (1991)]: (U, ω, θ), where $U \subseteq R^2$ is the path connected domain, ω is the holomorphic 1-form on U, θ is the meromorphic function on U.

For example, consider the Weierstrass representation $(U, f\,dw, \theta)$, where f is a holomorphic function, $\omega = f\,dw$, $w = u + iv$ is the complex coordinate on U. Then the equations

$$x = c_1 + 2\,\mathrm{Re}\int 1/2f(1-\theta^2)\,dw, \quad y = c_2 + 2\,\mathrm{Re}\int i/2f(1+\theta^2)\,dw, \quad z = c_3 + 2\,\mathrm{Re}\int f\theta\,dw$$

describe in a Euclidean space some minimal surface N, where (x, y, z) are the standard coordinates. Consider the tensor of stresses on U:

$$\sigma^{ik} = \begin{pmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} = -\sigma^{11} \end{pmatrix},$$

where $\sigma^{3i} = 0$, $i = 1, 2, 3$. We suppose that $\partial_k\sigma^{ik} = 0$. Since $\lambda^2 = e^{\psi} = |f|^2(1 + |\theta|^2)^2$ we want to assign the conformal weights so that:

$$\sigma'^{ik} = \lambda^{-4}\sigma^{ik} = |f|^{-4}(1+|\theta|^2)^{-4}\begin{pmatrix}\sigma^{11} & \sigma^{12}\\ \sigma^{21} & -\sigma^{11}\end{pmatrix}. \tag{30}$$

Under $Q = -(n+4)$ (see formula (12)):

$$e'_{ik} = \lambda^2 e_{ik} = |f|^2(1+|\theta|^2)^2\begin{pmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{pmatrix} \tag{31}$$

under $Q = -(n+2)$ (see formula (13)) we have:

$$e'_{ik} = e_{ik} = \begin{pmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{pmatrix}. \tag{32}$$

It is easy to understand that by means of eqns (30)–(32) we "lift" the solution from U to the minimal surface N.

3.4. *The plasticity theory*

We analyse the conformal anomalies of the biharmonic operator in the context of the plasticity theory. Consider the Prandle-Rice theory (Mises plasticity model) [see, e.g., Pluvinage (1989)]. Erie's function is a solution for the equation:

$\Delta^2 F +$ (the plasticity terms) $= 0$. The plasticity terms are:

$$\alpha\sigma_T^{N-1}\Delta^2 F + \alpha\left\{\frac{\partial^2\sigma_T^{N-1}}{\partial x^2}\cdot\left(\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}\right) + \frac{\partial^2\sigma_T^{N-1}}{\partial y^2}\cdot\left(\frac{\partial^2 F}{\partial y^2} - \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right) + 3\frac{\partial^2\sigma_T^{N-1}}{\partial x\partial y}\cdot\frac{\partial^2 F}{\partial x\partial y}\right\},$$

where (x, y) are Euclidean coordinates, σ_T is the yield limit: $\sigma_T^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2$, N is a coefficient, α is the technical constant [see Pluvinage (1989) for the notation]. Let us assume that

$$\frac{\partial^2\sigma_T^{N-1}}{\partial x^2} = \frac{\partial^2\sigma_T^{N-1}}{\partial y^2},\quad \frac{\partial^2\sigma_T^{N-1}}{\partial x\partial y} = 0,$$

i.e., $\sigma_T^{N-1} = A(x^2+y^2)+B$, where A and B are constants (for example: $A = 1, B = 0$). We have:

$$\frac{1}{2}(\sigma_T^{N-1}+\alpha^{-1})^{-1}\frac{\partial^2\sigma_T^{N-1}}{\partial x^2}\Delta F + \Delta^2 F = 0. \tag{33}$$

Consider eqn (27): from $\Delta'^2 F' = 0$ we obtain $\{4\gamma^b\gamma_b - 4\gamma^b\nabla_b - 2\nabla_b\gamma^b\}\Delta F + \Delta^2 F = 0$ $(P = 0, F' = F)$. If $\Delta F = \text{const} \neq 0$ (the Poisson's equation) and

$$4\gamma^b\gamma_b - 2\nabla_b\gamma^b = \frac{1}{2}(\sigma_T^{N-1}+\alpha^{-1})^{-1}\frac{\partial^2\sigma_T^{N-1}}{\partial x^2}$$

then eqns (27) and (33) are identical. Since $h = \lambda^2 g = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$, it follows that:

$$4\left(\left(\frac{\partial\ln\lambda}{\partial x}\right)^2 + \left(\frac{\partial\ln\lambda}{\partial y}\right)^2\right) - 2\Delta\ln\lambda = ((x^2+y^2)+\alpha^{-1})^{-1}. \tag{34}$$

Note that $2\Delta\ln\lambda = \lambda^2 R'$, where R' is the scalar curvature of the metric h. Consider the case

$\lambda = \lambda(r)$, where $r = \sqrt{x^2+y^2}$. We obtain from eqn (34):

$$4(\ln'\lambda)^2 - 2\left(\ln''\lambda + \frac{\ln'\lambda}{r}\right) = (r^2+\alpha^{-1})^{-1} \tag{35}$$

where the stroke denotes differentiation.

Suppose that R' is constant, i.e., a Gauss curvature $K = -2R' = -1$, 0 or 1, respectively. It is known [see Dubrovin *et al.* (1986)] that h can be written locally as:

$$h \equiv \lambda^2\delta = \frac{a^2}{(b+cr^2)^2}(\mathrm{d}x^2+\mathrm{d}y^2), \quad a, b, c$$

are the constants, i.e., $R' = -8cb/a^2$. Substituting λ into eqn (35), we have: $(16c^2r^2+8cb)(r^2+\alpha^{-1}) = (b+cr^2)^2$. Therefore, $c = b = 0$ and h is singular.

The conclusion: if the scalar curvature of the metric h is not a constant then the conformal anomaly of the biharmonic operator describes the plasticity effects.

4. EXAMPLES

4.1. *Lame's problem*

We consider an axially symmetric tension for an infinite domain U with the circular opening (the particular case of Lame's problem), $g = \mathrm{d}x^2+\mathrm{d}y^2$. The boundary conditions are: $\sigma_{rr} = -p_a$ under $r = a$, $\sigma_{rr} = 0$ under $r = \infty$, where σ_{rr}, $\sigma_{\varphi\varphi}$ are the physical components of the tensor of stresses in polar coordinates: $\sigma^{11} = \sigma_{rr}$, $\sigma^{22} = \sigma_{\varphi\varphi}r^{-2}$. We have [see Hahn (1985)]: $\sigma_{rr} = -p_a(ar^{-1})^2$, $\sigma_{\varphi\varphi} = p_a(ar^{-1})^2$. The trace tensor of stresses equals zero, therefore the equations of equilibrium are c.i.: $\nabla'_i\sigma'^{ik} = \nabla'_i(\lambda^{-4}\sigma^{ik}) = \lambda^{-4}\nabla_i\sigma^{ik}$. Since $h = f^*g = \lambda^2 g$, where f is some (bi)holomorphic transformation, it follows that the tensor $\sigma'^{ik} = \lambda^{-4}(f^*\sigma)^{ik}$ satisfies the equations of equilibrium for any domain V: $U = f(V)$. If we use the complex notations then: f: $w \mapsto z = f(w)$, where $z = x+iy$, $w = u+iv$, $\lambda^{-4} = |\mathrm{d}z/\mathrm{d}w|^4$. For example consider the inversion (i.e., V is a disk): f: $w \mapsto z = 1/w$, where $z = re^{i\varphi}$, $w = \rho e^{i\theta}$, $\lambda^2 = \rho^4$, $|a| = 1$, $h = \mathrm{d}u^2+\mathrm{d}v^2 = \mathrm{d}\rho^2+\rho^2\,\mathrm{d}\theta^2$. We have: $\sigma'^{11} = -p_a\rho^{-2}$, $\sigma'^{22} = p_a\rho^{-4}$; $\sigma'_{\rho\rho} = -p_a\rho^{-2}$, $\sigma_{\theta\theta} = p_a\rho^{-2}$, i.e., the solution has a singularity. It can be shown that: under $Q = -(n+4)$ $A'^{ijkn} = \rho^{-4}(\mu h^{ij}h^{kn}+v(h^{ik}h^{jn}+h^{in}h^{jk}))$ (V is the heterogeneous disk) and under $Q = -(n+2)$ $A'^{ikmn} = \mu h^{ij}h^{kn}+v(h^{ik}h^{jn}+h^{in}h^{jk})$ (V is the isotropic disk).

4.2. *Plane and cylinder*

Let $M = R^2-\{0\}$ be the plane without the beginning of coordinates, $g \equiv \delta$. If we consider the polar coordinates (r, φ) then $N = R^2-\{0\}$ is a cylinder with the metric $h \equiv \delta/r^2 = \mathrm{d}r^2/r^2+\mathrm{d}\varphi^2$. Consider the solution of Lame's problem from Section 4.1:

$$\sigma^{11} = -p_a a^2 r^{-2}, \quad \sigma^{22} = p_a a^2 r^{-4}.$$

After the transfer we obtain for the cylinder ($\lambda^2 = r^{-2}$):

$$\sigma'^{11} = -p_a a^2 r^2, \quad \sigma'^{22} = p_a a^2;$$

the physical components are:

$$\sigma'_{rr} = -p_a a^2, \quad \sigma'_{\varphi\varphi} = p_a a^2.$$

4.3. *Ellipsoid*

Consider an ellipsoid $x^2/a^2+y^2/a^2+z^2 = 1$, where (x, y, z) are Cartesian coordinates,

$a \geqslant 1$. After the stereographic projection we have:

$$x = \frac{2a^2 u}{u^2+v^2+a^2}, \quad y = \frac{2a^2 v}{u^2+v^2+a^2}, \quad z = 1 - \frac{2a^2}{u^2+v^2+a^2},$$

where (u, v) are the coordinates of the XOY-plane (the "equatorial" plane). In the polar coordinates ($r = \sqrt{u^2+v^2}$, $\varphi = \operatorname{arctg}(u/v)$) the metric of ellipsoid can be written as:

$$g = \frac{4a^2}{(r^2+a^2)^2}\left(\left(\frac{4r^2(1-a^2)}{(r^2+a^2)^2}+1\right)\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2\right).$$

We wish to find a transformation $\tilde{r} = \tilde{r}(r)$; $\tilde{\varphi} = \varphi$ such that $g = \lambda^2(\tilde{r}, \tilde{\varphi})\ (\mathrm{d}\tilde{r}^2 + \tilde{r}^2\,\mathrm{d}\tilde{\varphi}^2)$. It can be shown that

$$\tilde{r} = \exp\left(\int_{r_0}^{r} \frac{\sqrt{(y^2-a^2)^2+4y^2}}{y(y^2+a^2)}\,\mathrm{d}y\right),$$

where r_0 is a constant. Finally, we obtain:

$$\lambda^2 = \left(\frac{4a^2}{(r^2+a^2)^2}\left(\frac{4r^2(1-a^2)}{(r^2+a^2)^2}+1\right)\right)\cdot\left(\frac{\mathrm{d}\tilde{r}}{\mathrm{d}r}\right)^{-2}.$$

4.4. *Enepper surface*

We consider the Weierstrass representation $(C, \mathrm{d}w, w)$, where $\omega = \mathrm{d}w$, $\theta = w$, $w = u + iv$, C is a complex plane. The equations:

$$x = \rho\cos\varphi - \frac{\rho^3}{3}\cos 3\varphi, \quad y = -\rho\sin\varphi - \frac{\rho^3}{3}\sin 3\varphi, \quad z = \rho^2\cos 2\varphi$$

describe the Enepper surface [see Tuzilin and Fomenko (1991)], where (ρ, φ) are polar coordinates of the w-plane. Consider (on C) the field of stresses:

$$\sigma^{ik} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$$

where A, B are constants, $\sigma^{3j} = 0, j = 1, 2, 3$.

Thus the equations of equilibrium are true. If the w-plane is the isotropic solid then:

$$e^{ik} = \frac{1}{E}(1+v)\begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

where E is the Young's modulus, v is the Poisson's coefficient. In the orthotropic case we have:

$$e^{11} = \frac{1}{E_1}(1+v_{21})\sigma^{11} = \frac{1}{E_1}(1+v_{21})A, e^{22} = \frac{1}{E_2}(1+v_{12})\sigma^{22} = -\frac{1}{E_2}(1+v_{12})A;$$

$$e^{33} = \frac{1}{E_3}(v_{13}+v_{23})\sigma^{22} = -\frac{1}{E_3}(v_{13}+v_{23})A, e^{12} = \frac{1}{2\mu_{12}}\sigma^{12} = \frac{1}{2\mu_{12}}B.$$

E_i, v_{ik}, μ_{ik} are technical constants.

By means of formulas (10)–(16) and (30)–(32) we can now establish:

$$\lambda^2 = e^{\psi} = |f|^2(1+|\theta|^2)^2 = 1\cdot(1+|w|^2)^2 = (1+u^2+v^2)^2 = (1+\rho^2)^2,$$

$$\sigma'^{ik} = \lambda^{-4}\sigma^{ik} = (1+\rho^2)^{-4}\begin{pmatrix} A & B \\ B & -A \end{pmatrix};$$

under $Q = -(n+4)$ (eqn (12)):

$$e'_{ik} = \lambda^2 e_{ik} = (1+\rho^2)^2 \frac{1}{E}(1+\nu)\begin{pmatrix} A & B \\ B & -A \end{pmatrix},$$

under $Q = -(n+2)$ (eqn (13)):

$$e'_{ik} = e_{ik} = \frac{1}{E}(1+\nu)\begin{pmatrix} A & B \\ B & -A \end{pmatrix}.$$

In the orthotropic case we may calculate by analogy with the isotropic case. It is important to understand that we calculate the component e'^{33} by means of the formula

$$e'^{33} = \frac{1}{E'_3}(\nu'_{13}+\nu'_{23})\sigma'^{22},$$

where E'_3, ν'_{ik} are the technical constants after the conformal change of the metric. So, we "lift" the solution from R^2 to the Enepper surface. These calculations are correct for other minimal surfaces. For example: catenoid: Weierstrass representation $(C-\{0\}, \mathrm{d}w/2w^2, w)$; conformal multiplier λ^2

$$\frac{(1+\rho^2)^2}{4\rho^4}.$$

Richmond surface: $(C-\{0\},\ w^2\,\mathrm{d}w,\ 1/w^2)$; $\rho^4(1+\rho^{-4})^2$. Partial Scherk surface: $(U = \{|w| < 1\}, \mathrm{d}w/(1-w^4), w)$;

$$((1-u^4-v^4+6v^2u^2)^2+(4vu^3+4v^3u)^2)(1+\rho^2)^2.$$

Note that catenoid and helicoid are locally isometric.

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